

## CIRCULANT GRAPHS AND QUANTUM STATE TRANSFER

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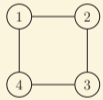
### 1 Introduction

Circulant graphs have a vast number of uses and applications in telecommunication networks, VLSI design, parallel and distributed computing. In this work we study circulant graphs that have integral eigen values. In particular, we bound the size of such an integral circulant graph in terms of its degree and give exact bounds for its diameter. We also study the transfer of quantum states on such networks.

### 2 Circulant Graphs

Let  $S = \{s_1, s_2, \dots, s_k\}$  be a set of  $k$  integers in the range  $\{1, 2, \dots, n\}$ . A *circulant graph*  $\mathcal{G} = G(n; S)$  is a graph on the set of  $n$  vertices  $V(\mathcal{G}) = \{v_1, \dots, v_n\}$  with an edge between  $v_i$  and  $v_j$  whenever  $|i - j| \in S$ .

For example,  $S = \{1, 3\}$  over the vertices  $\{1, 2, 3, 4\}$  gives the graph  $\mathcal{G}$ :



with adjacency matrix  $A_{\mathcal{G}} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$ .

#### 2.1 Integrality

We say that  $\mathcal{G}$  is *integral* if the adjacency matrix  $A_{\mathcal{G}}$  of  $\mathcal{G}$  has all integral eigen values. It is known that in this case there will be an  $F \subseteq [n]$  such that:

$$S = \bigcup_{f \in F} \left\{ i \mid 1 \leq i \leq n, \gcd(i, n) = \frac{n}{f} \right\}$$

#### 2.2 Degree and Size

The number of vertices  $n$  in a connected integral circulant graph cannot be arbitrarily large for a given degree  $k$ .

**Result 1:** We show that  $n = k^{O(\sqrt{k \log \log k})}$ .

#### 2.3 Diameter

The *diameter* of a graph  $\mathcal{G}$ , denoted by  $\text{diam}(\mathcal{G})$ , is the longest among the shortest paths between any two vertices. We show the following bound on the diameter of the integral, connected, circulant graph  $\mathcal{G} = G(n; S)$ :

**Result 2:** Let  $t$  be the size of the smallest set of additive generators of  $\mathbb{Z}_n$  contained in  $S$  then  $t \leq \text{diam}(\mathcal{G}) \leq 2t + 1$ .

We also show that the above bounds are tight by constructing graphs having diameters  $t$  and  $(2t + 1)$ .

### 3 Quantum Spin Systems

There are various physical systems that can serve as quantum channels, one of them being a quantum spin system. This can be generally defined as a collection of interacting qubits on a graph, whose dynamics is governed by a suitable Hamiltonian.

Let us define a quantum spin system by attaching a spin- $\frac{1}{2}$  particle to each of the  $n$  vertices of a graph  $\mathcal{G}$ . The Hilbert space associated with  $\mathcal{G}$  is given by  $\mathcal{H} \cong (\mathbb{C}^2)^{\otimes n}$ . This quantum system can be interpreted as a noiseless quantum channel and its dynamics can be seen as a continuous-time quantum walk on  $\mathcal{G}$  whose Hamiltonian is identical to the adjacency matrix of a graph  $\mathcal{G}$ .

We study two properties of quantum spin systems defined over circulant graphs.

#### 3.1 Periodicity

It is known that the dynamics of a quantum system is *periodic* if for every state  $|\psi\rangle \in \mathcal{H}$ , there exists  $t \in \mathbb{R}$ ,  $0 < t < \infty$ , for which  $|\langle \psi | e^{-iA_{\mathcal{G}}2t} | \psi \rangle| = 1$ . The number  $2t$  is the period of the system.

**Result 3:** We show that for a circulant  $\mathcal{G}$  the quantum spin system is periodic iff  $\mathcal{G}$  is integral.

#### 3.2 Perfect state transfer

We say that there is *perfect state transfer* in a quantum system on a graph  $\mathcal{G}$  if there are distinct vertices  $a, b$  and  $0 < t < \infty$ , such that:

$$|\langle a | e^{-iA_{\mathcal{G}}t} | b \rangle| = 1$$

This can be interpreted as having perfect communication between  $a$  and  $b$  in time  $t$ .

**Result 4:** We show that there is no perfect state transfer in circulant graphs.

However, the following questions remain **open**:

- Is there perfect state transfer in a circulant graph  $\mathcal{G}$  between a superposition of vertices?
- Circulant graphs are Cayley graphs over the group  $\mathbb{Z}_n$ . Is there perfect state transfer in a Cayley graph over other Abelian groups?