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Deflated PCG Method for the Poisson Solver

J.M. Tang, C. Vuik

Introduction

The Preconditioned Conjugate Gradient (PCG) method is applied to a large symmetric and positive semi-definite linear system $Ax = b$ coming from the Poisson equation

$$\text{div}\left(\frac{1}{\rho(\mathbf{x})}\nabla p(\mathbf{x})\right) = f(\mathbf{x}), \quad \mathbf{x} = (x, y, z) \in \Omega, \qquad (1)$$

with Neumann boundary conditions. Here, p is the pressure and ρ is the density.

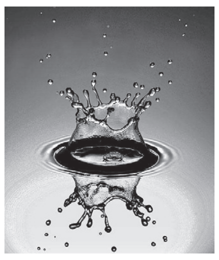

Figure 1: Our application of the Poisson equation: multi-phase flows. In this droplet splash, we have the phases air and water.

Due to large contrasts in ρ, matrix A is ill-conditioned. Appropriate preconditioners are not available, so that a deflation technique is used to accelerate the convergence of PCG.

Deflated PCG method

In Deflated PCG we solve

$$M^{-1}PA\tilde{x} = M^{-1}Pb. \qquad (2)$$

Here, M is the preconditioner and P is the deflation matrix,

$$P := I - AZE^{-1}Z^T, \quad E := Z^TAZ, \qquad (3)$$

with the identity matrix I and the deflation subspace matrix $Z = [z_1\ z_2\ \cdots\ z_r]$. We apply subdomain deflation, where Ω is divided in subdomains Ω_j independent of ρ and each subdomain corresponds to one sparse deflation vector consisting of ones and zeros.

Advantages of Deflated PCG:

- it converges faster than PCG;
- the extra costs are low;
- it is easy to implement in an existing code.

Example

The 3-D domain Ω consists of water with eight air-bubble, see also Figure 2.

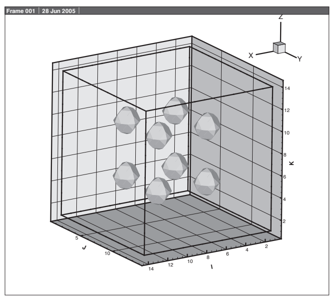

Figure 2: Geometry of a 3-D bubbly flow problem with eight air bubbles in the unit domain filled with water.

In this case, we have

$$\rho = \begin{cases} \rho_0 = 1, & x \in \Omega_0 \quad \text{(water)}, \\ \rho_1 = 10^{-3}, & x \in \Omega_1 \quad \text{(air)}. \end{cases} \qquad (4)$$

In order to solve $Ax = b$ with $100^3 = 1.000.000$ unknowns, PCG with the Incomplete Cholesky preconditioner (= ICCG) and its deflated variant (= DICCG) are applied, see Table 1.

Method	Defl. Vectors	# Iterations	CPU Time
ICCG	–	291	43.0
DICCG	8	160	29.1
	125	72	14.2
	1.000	**36**	**8.2**
	8.000	22	27.2

Table 1: Number of iterations and the CPU time (seconds) for ICCG and DICCG. A 3-D Poisson problem is considered on an unit domain with eight bubbles. Standard finite differences method with $n = 100^3$ is used.

It can observed that DICCG with 1.000 deflation vectors is the most efficient method. Compared to ICCG, it is 5.2 times faster in convergence.

Conclusion

A simple subdomain deflation technique clearly accelerates the convergence of PCG in order to solve the 3-D Poisson equation derived from multi-phase flows.

References

- J.M. Tang and C. Vuik, *On Deflation and Singular Symmetric Positive Semi-Definite Matrices*, JCAM, 2006, *to appear*.
- J.M. Tang and C. Vuik, *Efficient Deflation Method applied on 2-D and 3-D Bubbly Flow Problems*, ETNA, 2006, *submitted*.

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Email: j.m.tang@tudelft.nl

Web: http://ta.twi.tudelft.nl/nw/users/tang/

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