Asymptotic analysis for the busy period of the M/G/1 queue.

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Approximations and asymptotics

Very few network models can be analysed completely. In many applications, there is a need for estimating the very small probabilities of 'rare events'. Therefore, there is a need for approximation techniques and asymptotic methods. Besides, this also allows to obtain qualitative behaviour of the system when the 'rare event' happens. For the case of heavy-tailed distributions, recently obtained limit theorems and asymptotic expansions for heavy-tailed queues and random walks may be useful.

The single-server queue

Consider i.i.d. exponential interarrival times A_i and service times B_i. Assume that the system is stable, i.e. $\rho := \frac{\mathbf{E}B_1}{\mathbf{E}A_1} < 1$. Denote the number of customers arriving before time t by

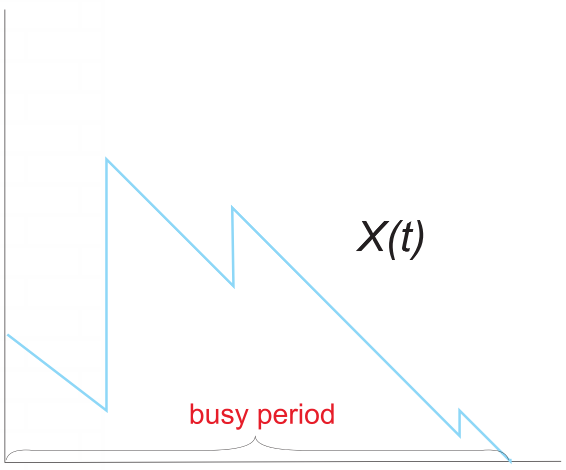
$$N(t) = \max\{n : A_1 + \ldots + A_n \leq t\}.$$

The unprocessed work at time t

$$X(t) = \sum_{i=0}^{N(t)} B_i - t.$$

Busy period of the M/G/1 queue

$$\tau = \inf\{t : X(t) < 0\}.$$



Heavy-tailed asymptotics

We say that ξ is heavy-tailed if $\mathbf{E}e^{\varepsilon\xi} = \infty$ for all $\varepsilon > 0$. In particular, the latter condition holds for regularly varying distributions (including Pareto) and Weibull(β) distributions $\mathbf{P}(\xi > x) = e^{-x^\beta}$ with parameter $\beta < 1$.

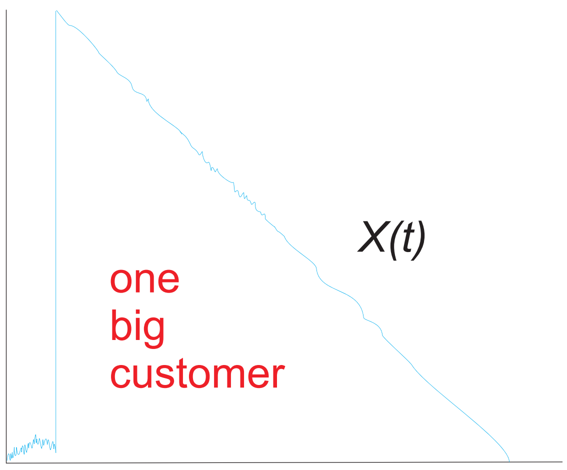
Known asymptotics for the busy period

De Meyer and Teugels (1980) obtain asymptotics for τ in the M/G/1 queue with regularly varying service times. Zwart (2001) considered the GI/GI/1 queue with regularly varying service times. Jelenković and Momčilović (2004), Baltrūnas, Daley and Klüppelberg (2004) considered the GI/GI/1 queue with Weibull (β) service times with $\beta < 1/2$. They show that for these distributions

$$\mathbf{P}(\tau > t) \sim \frac{1}{1-\rho}\mathbf{P}(B_1 > (1-\rho)t).$$

Typical event

In the above case the large busy period is most probably caused by some big customer arriving near time 0.



References

[1] Denisov and Shneer. Local asymptotics for the cycle maximum of a heavy-tailed random walk. (*to appear in Advances of Applied Probability*). EURANDOM report 2005-034.

[2]Denisov and Shneer. Asymptotics for first passage times of Levy processes and random walks. *Submitted.* EURANDOM report 2006-017.

[3]Denisov and Shneer. Global and local asymptotics for the busy period of the M/G/1 queue. *Submitted.* EURANDOM report 2007-001.

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Our results

First we reduce the original problem to the problem of large deviations for random walks.
Theorem. For heavy-tailed distributions under some natural conditions:

$$\mathbf{P}(\tau > t) \sim \frac{\mathbf{E}A_1}{1-\rho}\frac{\mathbf{P}(X(t) > 0)}{t}.$$

Using large deviations results for random walks:
Corollary 1. For regularly varying, lognormal, Weibull(β), $\beta < 1/2$,

$$\mathbf{P}(\tau > t) \sim \frac{1}{1-\rho}\mathbf{P}(B_1 > (1-\rho)t).$$

Corollary 2. For Weibull(β), $\beta \in (1/2, 1)$,

$$\mathbf{P}(\tau > t) \sim \frac{1}{1-\rho}\exp\{-((1-\rho)t)^\beta + \sum_{i=1}^{k} C_i t^{(i+1)\beta - i}\},$$

for certain k and constants $C_1, \ldots, C_k$.

It is also possible to find more detailed asymptotics $\mathbf{P}(\tau \in (t, t+T]), 0 < T < \infty, t \to \infty$, see [3] for details.

Typical event

The case of Weibull(β), $\beta \in (1/2, 1)$ service times differs qualitatively from the first case. Still this rare event is caused by some big customer arriving at the begining of the busy period, but now one should also take into account the influence of the 'normal' perturbations.